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On the image of the Burau representation of the IA-automorphism group of a free group

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ABSTRACT

In this paper we study the graded quotients of the lower central series of the image of the IA-automorphism group of a free group by the Burau representation. In particular, we determine their structures for degrees 1 and 2.

1. Introduction

For $n \geq 2$, let F_n be a free group of rank n with basis $x_1, x_2, \dots, x_n$, and $\Gamma_n(1) := F_n, \Gamma_n(2), \dots$ its lower central series. We denote by $\mathrm{Aut}\, F_n$ the group of automorphisms of F_n. For each $k \geq 0$, let $\mathcal{A}_n(k)$ be the group of automorphisms of F_n which induce the identity on the quotient group $F_n/\Gamma_n(k+1)$. Then we have a descending filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$, which is called the Johnson filtration of $\mathrm{Aut}\, F_n$. The Johnson filtration of $\mathrm{Aut}\, F_n$ was originally introduced in 1963 with the remarkable pioneer work by Andreadakis [1] who showed that $\mathcal{A}_n(1), \mathcal{A}_n(2), \dots$ is a central series of $\mathcal{A}_n(1)$, and that the graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank for each $k \geq 1$. Furthermore, he [1] also showed that $\mathcal{A}_2(1), \mathcal{A}_2(2), \dots$ coincides with the lower central series of $\mathcal{A}_2(1)$.

The group $\mathcal{A}_n(1)$ is called the IA-automorphism group which is also denoted by IA_n. Magnus [2] showed that IA_n is finitely generated. Furthermore, recently, Cohen–Pakianathan [3,4], Farb [5] and Kawazumi [6] independently determined the abelianization of IA_n. (See Section 2.2.) In general, however, the group structure of IA_n is far from being well understood. For example, a presentation of IA_n is still not known. For $n = 3$, Krstić and McCool [7] showed that IA_3 is not finitely presentable. For $n \geq 4$, it is not known whether IA_n is finitely presentable or not. In addition to this, even the structures of the low dimensional (co)homology of IA_n are not completely determined.

The Lie algebra with graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ plays an important role in understanding the group structure and cohomology of IA_n. In order to investigate each of $\mathrm{gr}^k(\mathcal{A}_n)$, certain injective homomorphisms

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

are defined. These homomorphisms are called the Johnson homomorphisms of $\mathrm{Aut}\, F_n$. (For definition, see [8,9].) Recently, the study of the Johnson filtration and the Johnson homomorphisms of $\mathrm{Aut}\, F_n$ achieved good progress through the work of many authors, for example, [3–6,10,11,8,12,9]. Here, we are interested in the following two problems. One is to determine whether $\mathcal{A}_n(k)$ coincides with the kth term $\mathcal{A}'_n(k)$ of the lower central series of $\mathrm{IA}_n = \mathcal{A}_n(1)$ or not. Andreadakis [1] showed that $\mathcal{A}_3(3) = \mathcal{A}'_3(3)$. Cohen–Pakianathan [3,4], Farb [5] and Kawazumi [6] independently showed that $\mathcal{A}_n(2) = \mathcal{A}'_n(2)$ for

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any $n \geq 3$. Furthermore, recently, Pettet [12] obtained that $\mathcal{A}'_n(3)$ has a finite index in $\mathcal{A}_n(3)$. However, it seems that there are few results for higher degrees. The other problem is to determine the abelianization of each $\mathcal{A}_n(k)$ for $k \geq 2$. From the study of the Johnson homomorphisms of $\mathrm{Aut}\,F_n$, we see that it contains a free abelian group of finite rank. However, it is not known even whether each of $H_1(\mathcal{A}_n(k), \mathbf{Z})$ is finitely generated or not.

In this paper, we study the images of $\mathcal{A}_n(k)$ and $\mathcal{A}'_n(k)$ through the Burau representation, which is one of the most important Magnus representations of $\mathrm{Aut}\,F_n$ defined on IA_n. (For definition, see Section 2.4.) In general, the Magnus representations of $\mathrm{Aut}\,F_n$ are representations of various subgroups of $\mathrm{Aut}\,F_n$ that make use of Fox's free differential calculus. (See [13] for details.) In this paper, we denote the Burau representation by τ_B, and write $\mathcal{B}_n(k) := \tau_B(\mathcal{A}_n(k))$ and $\mathcal{B}'_n(k) := \tau_B(\mathcal{A}'_n(k))$. First, we determine the abelianization of $\tau_B(\mathrm{IA}_n)$.

Theorem 1. *For any $n \geq 2$, $H_1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \cong \mathbf{Z}^{\oplus n(n-1)}$.*

Next, to study $\mathcal{B}'_n(k)$ and its graded quotients $\mathrm{gr}^k(\mathcal{B}'_n) := \mathcal{B}'_n(k)/\mathcal{B}'_n(k+1)$ for $k \geq 2$, we consider a certain normal subgroup of $\tau_B(\mathrm{IA}_n)$. For $1 \leq i \neq j \leq n$, let L_{ij} be an automorphism of F_n defined by

$$L_{ij} : \begin{cases} x_i \mapsto x_j x_i x_j^{-1}, \\ x_t \mapsto x_t, \quad (t \neq i). \end{cases}$$

We denote by $\mathcal{Y}_n$ a subgroup of $\tau_B(\mathrm{IA}_n)$ generated by L_{in} and L_{nj} for $1 \leq i, j \leq n-1$. Let $\mathcal{Y}'_n(k)$ be the lower central series of $\mathcal{Y}_n$. Then we prove:

Theorem 2. *For any $n \geq 2$ and $k \geq 2$, $\mathcal{Y}'_n(k) = \mathcal{B}'_n(k)$.*

Using this, we show:

Theorem 3. *For $n \geq 2$, $\mathrm{gr}^2(\mathcal{B}'_n) \cong \mathbf{Z}^{\oplus(n^2-n-1)}$.*

Observing the proof of the theorem above, as a corollary, we obtain:

Corollary 1. *For $n \geq 2$, $\mathcal{B}_n(3) = \mathcal{B}'_n(3)$.*

To show these, for $1 \leq l \leq k$, we define certain homomorphisms $\psi_{k,l}$ from $\mathcal{B}_n(k)$ to a free abelian group, and determine its image in Section 3. Using these homomorphisms, we detect a free abelian subgroup of $\mathrm{gr}^k(\mathcal{B}_n)$ and $\mathrm{gr}^k(\mathcal{B}'_n)$. We also show:

Corollary 2. *For $n \geq 2$, $k \geq 2$ and $1 \leq l \leq k$, $\psi_{k,l}(\mathcal{A}_n(k)) = \psi_{k,l}(\mathcal{A}'_n(k))$.*

This shows that the difference between $\mathcal{A}_n(k)$ and $\mathcal{A}'_n(k)$ is characterized by the kernel of the homomorphisms $\psi_{k,l}$. Furthermore, observing the image of $\psi_{k,k}$, we obtain:

Corollary 3. *For $n \geq 2$ and $k \geq 2$, $H_1(\mathcal{A}_n(k), \mathbf{Z}) \supset \mathbf{Z}^{\oplus k(n^2-n-1)}$.*

We remark that we cannot detect all of $\mathbf{Z}^{\oplus k(n^2-n-1)} \subset H_1(\mathcal{A}_n(k), \mathbf{Z})$ by the kth Johnson homomorphism of $\mathrm{Aut}\,F_n$ since some part of $\mathbf{Z}^{\oplus k(n^2-n-1)}$ is contained in $\mathcal{A}_n(k+1)$.

As an application, using a result $\mathrm{gr}^2(\mathcal{B}'_n) \cong \mathbf{Z}^{\oplus n^2-n-1}$, we can determine the image of the cup product $\cup : \Lambda^2 H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z})$. We show:

Theorem 4. *For $n \geq 2$, $\mathrm{Im}(\cup) \cong \mathbf{Z}^{\oplus(n-2)(n+1)(n^2-n-1)/2}$.*

Finally, we consider the case where $n = 2$. In particular, we show

Theorem 5. *For any $k \geq 2$, $\mathrm{gr}^k(\mathcal{B}'_2) \cong \mathbf{Z}$.*

Here we remark that by a result of Andreadakis [1], we have $\mathrm{gr}^k(\mathcal{B}_2) = \mathrm{gr}^k(\mathcal{B}'_2)$ for each $k \geq 1$.

In Section 2, we show the definition and some properties of the IA-automorphism group, the Johnson filtration and the Magnus representations of the automorphism group of a free group. In Section 3, to study the $\mathrm{gr}^k(\mathcal{B}_n)$ and $\mathrm{gr}^k(\mathcal{B}'_n)$, we define homomorphisms $\psi_{k,l}$ and determine their images. In Section 4, we consider the lower central series $\mathcal{B}'_n(k)$ of $\tau_B(\mathrm{IA}_n)$. In particular, we determine the structure of the graded quotients $\mathrm{gr}^k(\mathcal{B}'_n)$ for $k = 1$ and 2. In Section 5, we determine the image of the cup product map $\cup : \Lambda^2 H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z})$. Finally, In Section 6, we consider the case where $n = 2$.

2. Preliminaries

In this section, we recall the definition and some properties of the IA-automorphism group and the Magnus representations of the automorphism group of a free group.

2.1. Notation

Throughout the paper, we use the following notation and conventions.

- For a group G, the abelianization of G is denoted by G^{ab}.

- For a group G, the group $\mathrm{Aut}\,G$ acts on G from the right. For any $\sigma \in \mathrm{Aut}\,G$ and $x \in G$, the action of σ on x is denoted by x^σ.
- For a group G, and its quotient group G/N, we also denote the coset class of an element $g \in G$ by $g \in G/N$ if there is no confusion.
- For elements x and y of a group, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

2.2. IA-automorphism group

For $n \geq 2$, let F_n be a free group of rank n with basis $x_1, \ldots, x_n$. We denote the abelianization of F_n by H, and its dual group by $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$. Let $\rho : \mathrm{Aut}\,F_n \to \mathrm{Aut}\,H$ be the natural homomorphism induced from the abelianization of F_n. In this paper we identify $\mathrm{Aut}\,H$ with the general linear group $\mathrm{GL}(n, \mathbf{Z})$ by fixing the basis of H as a free abelian group induced from the basis $x_1, \ldots, x_n$ of F_n. The kernel IA_n of ρ is called the IA-automorphism group of F_n. It is well known due to Nielsen [14] that IA_2 coincides with the inner automorphism group $\mathrm{Inn}\,F_2$ of F_2. Namely, IA_2 is a free group of rank 2. However, IA_n for $n \geq 3$ is much larger than the inner automorphism group $\mathrm{Inn}\,F_n$ of F_n. Indeed, Magnus [2] showed that for any $n \geq 3$, IA_n is finitely generated by automorphisms

$$K_{ij} : x_t \mapsto \begin{cases} x_j^{-1} x_i x_j, & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct $i, j \in \{1, 2, \ldots, n\}$ and

$$K_{ijk} : x_t \mapsto \begin{cases} x_i [x_j, x_k], & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct $i, j, k \in \{1, 2, \ldots, n\}$ such that $j < k$. In this paper, for convenience, we often use automorphisms $L_{ij} := K_{ij}^{-1}$ and $L_{ijk} := K_{ijk}[K_{ij}^{-1}, K_{ik}^{-1}]$. Then we see that

$$L_{ij} : x_t \mapsto \begin{cases} x_j x_i x_j^{-1}, & t = i, \\ x_t, & t \neq i, \end{cases} \qquad L_{ijk} : x_t \mapsto \begin{cases} [x_j, x_k] x_i, & t = i, \\ x_t, & t \neq i, \end{cases}$$

and that IA_n is also generated by L_{ij} and L_{ijk}. Recently, Cohen–Pakianathan [3,4], Farb [5] and Kawazumi [6] independently showed

$$\mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H \tag{1}$$

as a $\mathrm{GL}(n, \mathbf{Z})$-module.

2.3. Johnson filtration

In this subsection we briefly recall the definition and some properties of the Johnson filtration of $\mathrm{Aut}\,F_n$. (For details, see [9] for example.)

Let $\Gamma_n(1) \supset \Gamma_n(2) \supset \cdots$ be the lower central series of a free group F_n defined by

$$\Gamma_n(1) := F_n, \qquad \Gamma_n(k) := [\Gamma_n(k-1), F_n], \quad k \geq 2.$$

For $k \geq 0$, the action of $\mathrm{Aut}\,F_n$ on each nilpotent quotient $F_n/\Gamma_n(k+1)$ induces a homomorphism

$$\rho^k : \mathrm{Aut}\,F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1)).$$

The map ρ^0 is trivial, and $\rho^1 = \rho$. We denote the kernel of ρ^k by $\mathcal{A}_n(k)$. Then the groups $\mathcal{A}_n(k)$ define a descending central filtration

$$\mathrm{Aut}\,F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\,F_n$, with $\mathcal{A}_n(1) = IA_n$. We call it the Johnson filtration of $\mathrm{Aut}\,F_n$, and denote each of its graded quotient by $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$.

The Johnson filtration of $\mathrm{Aut}\,F_n$ was originally introduced in 1963 in the remarkable pioneer work by Andreadakis [1] who showed that $\mathcal{A}_n(1), \mathcal{A}_n(2), \ldots$ is a descending central series of $\mathcal{A}_n(1)$ and $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank. The Johnson filtration has been studied with the Johnson homomorphisms of $\mathrm{Aut}\,F_n$. The study of the Johnson homomorphisms was begun in 1980 by D. Johnson [15]. He [16] studied the Johnson homomorphism of a mapping class group of a closed oriented surface, and determined the abelianization of the Torelli group. The Johnson homomorphisms of $\mathrm{Aut}\,F_n$ are also defined in a similar way, and there is a broad range of remarkable results for them. (For surveys and related topics concerning with the Johnson homomorphisms, see [11,8] for example.)

Let $\mathcal{A}'_n(1), \mathcal{A}'_n(2), \ldots$ be the lower central series of IA_n. In this paper, we are interested in the difference between $\mathcal{A}_n(k)$ and $\mathcal{A}'_n(k)$. Andreadakis [1] showed that the filtration $\mathcal{A}_2(1), \mathcal{A}_2(2), \ldots$ coincides with the lower central series of $\mathcal{A}_2(1) = \mathrm{Inn}\,F_2$, and that $\mathcal{A}_3(3) = \mathcal{A}'_3(3)$. Recently, Cohen–Pakianathan [3,4], Farb [5] and Kawazumi [6] independently showed that $\mathcal{A}_n(2) = \mathcal{A}'_n(2)$ for any $n \geq 3$. Pettet [12] showed that $\mathcal{A}'_n(3)$ has a finite index in $\mathcal{A}_n(3)$ at most for any $n \geq 3$. In general, however, it is still an open problem whether the Johnson filtration $\mathcal{A}_n(1), \mathcal{A}_n(2), \ldots$ coincides with the lower central series of IA_n or not.

2.4. Magnus representations

In this subsection we recall the Magnus representation of IA_n. (For details, see [13].) For each $1 \le i \le n$, let

$$\frac{\partial}{\partial x_i} : \mathbf{Z}[F_n] \to \mathbf{Z}[F_n]$$

be Fox's derivation defined by

$$\frac{\partial}{\partial x_i}(w) = \sum_{j=1}^{r} \epsilon_j \delta_{\mu_j, i} x_{\mu_1}^{\epsilon_1} \cdots x_{\mu_j}^{\frac{1}{2}(\epsilon_j - 1)} \in \mathbf{Z}[F_n]$$

for any reduced word $w = x_{\mu_1}^{\epsilon_1} \cdots x_{\mu_r}^{\epsilon_r} \in F_n$, $\epsilon_j = \pm 1$. (For details for Fox's derivation, see [17].) Let $\varphi : F_n \to G$ be any group homomorphism. If there is no confusion, we also denote by φ both the ring homomorphism $\bar{\varphi} : \mathbf{Z}[F_n] \to \mathbf{Z}[G]$ induced from φ and the group homomorphism $\hat{\varphi} : \mathrm{GL}(n, \mathbf{Z}[F_n]) \to \mathrm{GL}(n, \mathbf{Z}[G])$ induced from $\bar{\varphi}$. For any matrix $C = (c_{ij}) \in \mathrm{GL}(n, \mathbf{Z}[F_n])$, let C^{φ} be the matrix $(c_{ij}^{\varphi}) \in \mathrm{GL}(n, \mathbf{Z}[G])$. Then we obtain a map $\tau_{\varphi} : \mathrm{Aut}\, F_n \to \mathrm{GL}(n, \mathbf{Z}[G])$ defined by

$$\sigma \mapsto \left(\frac{\partial x_i{}^{\sigma}}{\partial x_j} \right)^{\varphi}.$$

This map is not a homomorphism in general. Let A_{φ} be a subgroup of $\mathrm{Aut}\, F_n$ consisting of automorphisms σ such that $(x^{\sigma})^{\varphi} = x^{\varphi}$. Then, by restricting τ_{φ} to A_{φ}, we obtain a homomorphism

$$\tau_{\varphi} : A_{\varphi} \to \mathrm{GL}(n, \mathbf{Z}[G]),$$

which is called the Magnus representation of A_{φ}.

Here we consider two particular homomorphisms from F_n. The first one is the abelianization $\mathfrak{a} : F_n \to H$ of F_n. It is clear that $\mathrm{IA}_n \subset A_{\mathfrak{a}}$. We call the Magnus representation $\tau_{\mathfrak{a}} : \mathrm{IA}_n \to \mathrm{GL}(n, \mathbf{Z}[H])$ the Gassner representation of IA_n, denoted by τ_G. Let $s_1, \ldots, s_n$ be the coset classes of $x_1, \ldots, x_n$ in H respectively. Then, for example, $\tau_G(L_{ij})$ and $\tau_G(L_{ijk})$ are given by

$$\begin{array}{cc} & \begin{array}{cccc} & i & j & \end{array} \\ \begin{array}{c} \\ i \\ j \\ \\ \end{array} & \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & s_j & 1 - s_i & \vdots \\ \vdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{cc} & \begin{array}{ccccc} & i & j & k & \end{array} \\ \begin{array}{c} \\ i \\ j \\ k \\ \\ \end{array} & \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 - s_k & s_j - 1 & 0 \\ \vdots & 0 & 1 & 0 & 0 \\ \vdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \end{array}$$

respectively. Bachmuth determined the image $\mathrm{Im}(\tau_G)$ of τ_G:

Theorem 2.1 (*Bachmuth, [18]*). *For $n \ge 2$ and $C = (c_{ij}) \in \mathrm{GL}(n, \mathbf{Z}[H])$, $C \in \mathrm{Im}(\tau_G)$ if and only if C satisfies*

(1) $\det(C) = s_1^{e_1} s_2^{e_2} \cdots s_n^{e_n}$, $e_i \in \mathbf{Z}$,
(2) *For any* $1 \le i \le n$,

$$\sum_{j=1}^{n} c_{ij}(1 - s_j) = 1 - s_i.$$

Let $I := \mathrm{Ker}(\mathbf{Z}[F_n] \to \mathbf{Z})$ be the augmentation ideal of the group ring $\mathbf{Z}[H]$. By a fundamental argument in Fox's free differential calculus, we see that for any $C = (c_{ij}) \in \mathrm{Im}(\tau_G|_{\mathcal{A}_n(k)})$, $c_{ij} - \delta_{ij} \in I^k$ for any $i \ne j$. Here δ_{ij} is Kronecker's delta.

Let $\langle s \rangle$ be the infinite cyclic group generated by s. The other homomorphism is $\mathfrak{b} : F_n \to \langle s \rangle$ defined by $x_i \mapsto s$. The group ring $\mathbf{Z}[\langle s \rangle]$ is naturally considered as the Laurent polynomial ring $\mathbf{Z}[s^{\pm 1}]$ of one indeterminates over the integers. In this paper we identify them. Then we call the Magnus representation

$$\tau_B := \tau_{\mathfrak{b}} : \mathrm{IA}_n \to \mathrm{GL}(n, \mathbf{Z}[s^{\pm 1}]),$$

the Burau representation of IA_n. For a homomorphism $\mathfrak{c} : H \to \langle s \rangle$ defined by $s_i \mapsto s$, $\tau_B = \mathfrak{c} \circ \tau_G$. By Theorem 2.1, we have:

Lemma 2.1. *For $n \ge 2$, any element $C = (c_{ij}) \in \mathrm{Im}(\tau_B)$ satisfies*

(1) $\det(C) = s^e$, $e \in \mathbf{Z}$,
(2) *For any* $1 \le i \le n$,

$$\sum_{j=1}^{n} c_{ij} = 1.$$

Let $\mathcal{B}_n(k)$ and $\mathcal{B}_n'(k)$ be the images of $\mathcal{A}_n(k)$ and $\mathcal{A}_n'(k)$ by the Burau representation τ_B respectively. Let $J := \mathrm{Ker}(\mathbf{Z}[s^{\pm 1}] \to \mathbf{Z})$ be the augmentation ideal of the group ring $\mathbf{Z}[s^{\pm 1}]$. For any $k \ge 1$, an ideal J^k is a principal ideal generated by $(1 - s)^k$.

For any $C = (c_{ij}) \in \mathcal{B}_n(k)$, take an element $\sigma \in \mathcal{A}_n(k)$ such that $\tau_B(\sigma) = C$. Let

$$\pi : \mathrm{GL}(n, \mathbf{Z}[H]) \to \mathrm{GL}(n, \mathbf{Z}[s^{\pm 1}])$$

be a homomorphism induced by a homomorphism $H \to \langle s \rangle$, $s_i \mapsto s$, then we have $C = \pi \circ \tau_G(\sigma)$. If we set $\tau_G(\sigma) := (a_{ij}) \in \mathrm{GL}(n, \mathbf{Z}[H])$, we have $a_{ij} - \delta_{ij} \in I^k$ as above, and hence $c_{ij} - \delta_{ij} \in J^k$.

3. Homomorphisms $\psi_{k,l}$

In this section we study homomorphisms from subgroups of $\mathrm{GL}(n, \mathbf{Z}[s^{\pm 1}])$ to certain free abelian groups. The results, obtained in this section, are applied to determine the structure of the graded quotients $\mathrm{gr}^k(\mathcal{B}_n') := \mathcal{B}_n'(k)/\mathcal{B}_n'(k+1)$ for $k = 1$ and 2 in the next section.

For any $n \geq 2$ and $k \geq 1$, let $\Gamma(n, k)$ be the kernel of a homomorphism $\mathrm{GL}(n, \mathbf{Z}[s^{\pm 1}]) \to \mathrm{GL}(n, \mathbf{Z}[s^{\pm 1}]/J^k)$ induced from a natural projection $\mathbf{Z}[s^{\pm 1}] \to \mathbf{Z}[s^{\pm 1}]/J^k$. From the argument above, we see $\mathcal{B}_n(k) \subset \Gamma(n, k)$. We denote by $M(n, R)$ the abelian group of $(n \times n)$-matrices over a ring R. For any $k \geq 1$ and $1 \leq l \leq k$, we consider a map $\xi_{k,l} : \Gamma(n, k) \to M(n, \mathbf{Z}[s^{\pm 1}]/J^l)$ defined by

$$\xi_{k,l}(C) = C' \bmod J^l$$

where $C = E + (1-s)^k C'$, and E denotes the identity matrix. The map $\xi_{k,l}$ is a homomorphism since

$$(E + (1-s)^k C')(E + (1-s)^k D') = E + (1-s)^k (C' + D' + (1-s)^k C'D')$$

for any $C = E + (1-s)^k C'$, $D = E + (1-s)^k D' \in \Gamma(n, k)$. Set

$$\psi_{k,l} := \xi_{k,l} \circ \tau_B|_{\mathcal{A}_n(k)} : \mathcal{A}_n(k) \to M(n, \mathbf{Z}[s^{\pm 1}]/J^l).$$

In the following, we completely determine the image of $\psi_{k,l}$. First, we consider the case where $k = l = 1$.

Proposition 3.1. *For $n \geq 2$, $\mathrm{Im}(\psi_{1,1}) \cong \mathbf{Z}^{\oplus n(n-1)}$.*

Proof. We recall that IA_n is generated by L_{ij} and L_{ijk}. Since $\tau_B(L_{ijk}) = \tau_B(L_{ij} L_{ik}^{-1})$, $\mathrm{Im}(\psi_{1,1})$ is generated by the elements

$$\psi_{1,1}(L_{ij}) = \begin{array}{c} \\ \underline{i} \\ \\ \underline{j} \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} & \underline{i} & \underline{j} & \end{array} \\ \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & -1 & 1 & \vdots \\ \vdots & 0 & 0 & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \end{array}.$$

It is clear that $\psi_{1,1}(L_{ij})$, $1 \leq i \neq j \leq n$, is linearly independent over $\mathbf{Z}$. Hence $\mathrm{Im}(\psi_{1,1}) \cong \mathbf{Z}^{\oplus n(n-1)}$. $\square$

Now, for any $l \geq 1$, the quotient ring $\mathbf{Z}[s^{\pm 1}]/J^l$ is a free abelian group of rank l with a basis $\{(1-s)^m \mid 0 \leq m \leq l-1\}$. We fix this basis in the following. To study $\mathrm{Im}(\psi_{k,l})$ for $k \geq 2$, we consider some elements in $\mathcal{A}_n(k)$. For $k \geq 2$, $1 \leq l \leq k$ and $0 \leq m \leq l-1$, and distinct i, j and u, set

$$\sigma_m(i, j, u) := [L_{iju}, L_{ij}, L_{ij}, \ldots, L_{ij}] \in \mathcal{A}_n'(m+k) \subset \mathcal{A}_n(k)$$

where L_{ij} appears $m + k - 1$ times among the component. Then we see

$$\sigma_m(i, j, u) : x_t \mapsto \begin{cases} [x_j, x_u, x_j, x_j, \ldots, x_j] x_i, & t = i \\ x_t, & t \neq i \end{cases}$$

and

$$\psi_{k,l}(\sigma_m(i, j, u)) = \begin{array}{c} \\ \underline{i} \\ \\ \underline{j} \\ \\ \underline{u} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} & \underline{i} & \underline{j} & \underline{u} & \end{array} \\ \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & (1-s)^m & -(1-s)^m & \vdots \\ \vdots & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0 & 0 & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \end{array}.$$

For $0 \leq m \leq l-1$, and distinct i and j, set

$$w_m(i, j) := [K_{ij}, K_{ji}, K_{ij}, K_{ij}, \ldots, K_{ij}]^{-1} \in \mathcal{A}_n'(m+k) \subset \mathcal{A}_n(k)$$

where K_{ij} appears $m+k-2$ times among the component. Then we see

$$w_m(i,j) : x_t \mapsto \begin{cases} [x_i, x_j, x_j, \ldots, x_j, x_t]x_t, & t = i, j, \\ x_t, & t \neq i, j \end{cases}$$

and

$$\psi_{k,l}(w_m(i,j)) = \begin{array}{c} \\ \\ \underline{i} \\ \\ \underline{j} \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} \quad & \underline{i} & \underline{j} & \quad \end{array} \\ \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & (1-s)^m & -(1-s)^m & \vdots \\ \vdots & (1-s)^m & -(1-s)^m & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \end{array}.$$

Set

$$\begin{aligned} \mathfrak{E} := {} & \{\psi_{k,l}(\sigma_m(i,j,n)) \mid 1 \leq j < i \leq n-1, 0 \leq m \leq l-1\} \\ & \cup \{\psi_{k,l}(\sigma_m(n, n-1, u)) \mid 1 \leq u \leq n-2, 0 \leq m \leq l-1\} \\ & \cup \{\psi_{k,l}(w_m(i,j)) \mid 1 \leq i < j \leq n, 0 \leq m \leq l-1\} \subset \mathrm{Im}(\psi_{k,l}). \end{aligned}$$

Then we see:

Proposition 3.2. *For $n \geq 2$, $k \geq 2$ and $1 \leq l \leq k$, $\mathrm{Im}(\psi_{k,l})$ is a free abelian group with basis $\mathfrak{E}$. In particular, $\mathrm{Im}(\psi_{k,l}) \cong \mathbf{Z}^{\oplus l(n^2-n-1)}$.*

Proof. First, we show that $\mathfrak{E}$ generates $\mathrm{Im}(\psi_{k,l})$. For any $\sigma \in \mathcal{A}_n(k)$, set $\psi_{k,l}(\sigma) := (a_{ij}) \in M(n, \mathbf{Z}[s^{\pm 1}]/J^l)$, and set

$$a_{ij} = a_{ij}(0) + a_{ij}(1)(1-s) + \cdots a_{ij}(l-1)(1-s)^{l-1}$$

for $a_{ij}(0), \ldots, a_{ij}(l-1) \in \mathbf{Z}$. Then we have

$$\psi_{k,l}(\sigma w_0(1,2)^{a_{12}(0)} w_1(1,2)^{a_{12}(1)} \cdots w_{l-1}(1,2)^{a_{12}(l-1)}) = \begin{pmatrix} a_{11}+a_{12} & 0 & a_{13} & \cdots & a_{1n} \\ a_{21}+a_{12} & a_{22}-a_{12} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

Similarly, considering $w_m(1,j)$ for $0 \leq m \leq l-1$ and $3 \leq j$, we can transform $\psi_{k,l}(\sigma)$ to

$$\begin{pmatrix} a_{11}+a_{12}+\cdots a_{1n} & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & * \end{pmatrix}.$$

Using (2) of Lemma 2.1, we obtain $a_{11}+a_{12}+\cdots a_{1n} = 0$. Therefore, we may assume $a_{11} = a_{12} = \cdots = a_{1n} = 0$.

Next, considering $\sigma_m(2, 1, n)$, we see

$$\psi_{k,l}(\sigma \sigma_0(2,1,n)^{-a_{21}(0)} \sigma_1(2,1,n)^{-a_{21}(1)} \cdots \sigma_{l-1}(2,1,n)^{-a_{21}(l-1)}) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ * & * & \vdots & * \\ * & * & \cdots & * \end{pmatrix}.$$

Then using $w_m(2,j)$ for $0 \leq m \leq l-1$ and $3 \leq j$, we can transform this matrix to

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & * \end{pmatrix}.$$

Hence, we may assume $a_{21} = a_{22} = \cdots = a_{2n} = 0$. By repeating these processes, we see that the matrix $\psi_{k,l}(\sigma)$ is transformed to

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Furthermore, using $\sigma_m(n, n-1, u)$ for $1 \le u \le n-2$ and $0 \le m \le l-1$, we may assume

$$\psi_{k,l}(\sigma) = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & a_{nn-1} & a_{nn} \end{pmatrix}.$$

On the other hand, we have $\tau_B(\sigma) \equiv E + (1-s)^k \psi_{k,l}(\sigma) \bmod J^{k+l}$. Hence

$$1 = \det(\tau_B(\sigma)) \equiv 1 + (1-s)^k a_{nn} \bmod J^{k+l},$$

and

$$\begin{aligned} 0 &\equiv (1-s)^k a_{nn} \\ &= a_{nn}(0)(1-s)^k + a_{nn}(1)(1-s)^{k+1} + \cdots + a_{nn}(l-1)(1-s)^{k+l-1} \bmod J^{k+l}. \end{aligned}$$

This shows that $a_{nn}(0) = \cdots = a_{nn}(l-1) = 0$. Namely, $a_{nn} = 0$ and $a_{nn-1} = 0$. Therefore we conclude that $\mathfrak{E}$ generates $\mathrm{Im}(\psi_{k,l})$.

Finally, we show that the elements of $\mathfrak{E}$ are linearly independent. Suppose

$$\begin{aligned} &\sum_{0\le m\le l-1} \sum_{1\le j<i\le n-1} b_m(i,j,n)\psi_{k,l}(\sigma_m(i,j,n)) + \sum_{0\le m\le l-1} \sum_{1\le u\le n-2} b_m(n,n-1,u)\psi_{k,l}(\sigma_m(n,n-1,u)) \\ &\quad + \sum_{0\le m\le l-1} \sum_{1\le i<j\le n} c_m(i,j)\psi_{k,l}(w_m(i,j)) = 0 \end{aligned}$$

for integers $b_m(i,j,n)$, $b_m(n, n-1, u)$ and $c_m(i,j)$. Observing $(1,j)$-entry for $2 \le j$, we see $c_m(1,j) = 0$. Similarly, we obtain $b_m(2,1,n) = 0$ from $(2,1)$-entry, and $c_m(2,j) = 0$ from $(2,j)$-entry for $3 \le j$. By an inductive argument, we obtain $b_m(i,j,n) = 0$ and $c_m(i,j) = 0$. Finally, observing (n,u)-entry, we obtain $b_m(n,n-1,u) = 0$. Therefore we conclude that the elements of $\mathfrak{E}$ are linearly independent. This completes the proof of Proposition 3.2. □

From the proof of the Propositions above, we see:

Corollary 3.1. *For $n \ge 2$, $k \ge 2$ and $1 \le l \le k$, $\psi_{k,l}(\mathcal{A}_n(k)) = \psi_{k,l}(\mathcal{A}'_n(k))$.*

This shows that the difference between $\mathcal{A}_n(k)$ and $\mathcal{A}'_n(k)$ is characterized by the kernel of $\psi_{k,l}$. Furthermore, observing the image of $\psi_{k,k}$, we have:

Corollary 3.2. *For $n \ge 2$ and $k \ge 2$, $H_1(\mathcal{A}_n(k), \mathbf{Z})$ contains a free abelian group of rank $k(n^2-n-1)$.*

We also remark that this corollary gives a lower bound for the number of generators of $\mathcal{A}_n(k)$.

4. Filtration $\mathcal{B}'_n(k)$

In this section, we consider the lower central series $\mathcal{B}'_n(k)$ of $\mathcal{B}'_n(1) := \tau_B(\mathrm{IA}_n)$. In particular, we determine the structure of the graded quotients $\mathrm{gr}^k(\mathcal{B}'_n) := \mathcal{B}'_n(k)/\mathcal{B}'_n(k+1)$ for $k = 1$ and 2, using the homomorphisms $\xi_{1,1}$ and $\xi_{2,1}$. We also show that $\mathcal{B}_n(3) = \mathcal{B}'_n(3)$. First, we consider the case where $k = 1$, namely, the abelianization of $\tau_B(\mathrm{IA}_n)$.

Theorem 4.1. *For any $n \ge 2$, $H_1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \cong \mathbf{Z}^{\oplus n(n-1)}$.*

Proof. Since $\tau_B(L_{ijk}) = \tau_B(L_{ij}L_{ik}^{-1})$, $\tau_B(\mathrm{IA}_n)$ is generated by $\tau_B(L_{ij})$. In particular, $H_1(\tau_B(\mathrm{IA}_n), \mathbf{Z})$ is generated by $n(n-1)$ elements. On the other hand, the surjective homomorphism $\xi_{1,1} : \tau_B(\mathrm{IA}_n) \to \mathbf{Z}^{\oplus n(n-1)}$ induces a split surjective homomorphism $H_1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to \mathbf{Z}^{\oplus n(n-1)}$. Therefore, $H_1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \cong \mathbf{Z}^{\oplus n(n-1)}$. □

To study the graded quotients $\mathrm{gr}^k(\mathcal{B}'_n)$ for $k \ge 2$, we consider a certain normal subgroup $\mathcal{Y}_n$ of $\tau_B(\mathrm{IA}_n)$. Let $\mathcal{Y}_n$ be a subgroup of $\tau_B(\mathrm{IA}_n)$ generated by $\overline{L}_{in}$ and $\overline{L}_{nj}$ for $i, j \ne n$. In particular, we show that the lower central series $\mathcal{Y}'_n(k)$ of $\mathcal{Y}_n$ coincides with $\mathcal{B}'_n(k)$ for any $k \ge 2$. In the following, we use $\overline{L}_{ij}$ for $\tau_B(L_{ij})$ for simplicity.

Lemma 4.1. *For any $n \ge 2$, $\mathcal{Y}_n$ is a normal subgroup of $\tau_B(\mathrm{IA}_n)$.*

Proof. It suffices to show that

$$\overline{L}_{pq}^{\pm1}\overline{L}_{in}\overline{L}_{pq}^{\mp1}, \overline{L}_{pq}^{\pm1}\overline{L}_{nj}\overline{L}_{pq}^{\mp1} \in \mathcal{Y}_n$$

for any $\overline{L}_{pq}$.

Set $X := \overline{L}_{pq}^{\pm1}\overline{L}_{in}\overline{L}_{pq}^{\mp1}$ and $N := \sharp\{p, q, i, n\}$. If $N = 4$, we have $X = \overline{L}_{in} \in \mathcal{Y}_n$. For $N = 3$, there are four cases:

(i) $p = i$, $X = \overline{L}_{nq}^{\mp1}\overline{L}_{in}\overline{L}_{nq}^{\pm1} \in \mathcal{Y}_n$.

(ii) $p = n, X = \overline{L}_{nq}^{\pm 1}\overline{L}_{in}\overline{L}_{nq}^{\mp 1} \in \mathcal{Y}_n$.
(iii) $q = i, X = \overline{L}_{in}\overline{L}_{pn}\overline{L}_{ni}^{\mp 1}\overline{L}_{pn}^{-1}\overline{L}_{ni}^{\pm 1} \in \mathcal{Y}_n$.
(iv) $q = n, X = \overline{L}_{pn}^{\pm 1}\overline{L}_{in}\overline{L}_{pn}^{\mp 1} \in \mathcal{Y}_n$.

If $N = 2$, it is clear that $X \in \mathcal{Y}_n$. Similarly, set $X' := \overline{L}_{pq}^{\pm 1}\overline{L}_{nj}\overline{L}_{pq}^{\mp 1}$ and $N' := \sharp\{p, q, j, n\}$. For $N' = 4$ or 2, we see $X' \in \mathcal{Y}_n$. For $N = 3$, we have

(i) $p = j, X' = \overline{L}_{nq}^{\mp 1}\overline{L}_{nj}^{-1}\overline{L}_{nq}^{\pm 1} \in \mathcal{Y}_n$.
(ii) $p = n, X' = \overline{L}_{nq}^{\pm 1}\overline{L}_{nj}\overline{L}_{nq}^{\mp 1} \in \mathcal{Y}_n$.
(iii) $q = j, X' = \overline{L}_{nj} \in \mathcal{Y}_n$.
(iv) $q = n, X' = \overline{L}_{pn}^{\pm 1}\overline{L}_{nj}\overline{L}_{pn}^{\mp 1} \in \mathcal{Y}_n$.

This completes the proof of Lemma 4.1. □

From this lemma, we see that the natural action of $\tau_B(\mathrm{IA}_n)$ on $H_1(\mathcal{Y}_n, \mathbf{Z})$ by conjugation is trivial. Next, in order to show that $\mathcal{Y}_n$ contains the commutator subgroup of $\tau_B(\mathrm{IA}_n)$, we prepare some lemmas.

Lemma 4.2. *For $1 \leq i \neq j \leq n$, $[\overline{L}_{ij}, \overline{L}_{ji}] \in \mathcal{Y}_n$.*

Proof. It suffices to consider the case where $1 \leq i, j \leq n - 1$. In IA_n, we have

$$[K_{ji}K_{ni}, K_{ij}] = [K_{nij}, \ (K_{in}K_{jn})^{-1}],$$

hence,

$$[\overline{K}_{ji}\overline{K}_{ni}, \overline{K}_{ij}] = [\overline{K}_{ni}^{-1}\overline{K}_{nj}[\overline{K}_{nj}^{-1}, \overline{K}_{ni}^{-1}], \ (\overline{K}_{in}\overline{K}_{jn})^{-1}] \tag{2}$$

in $\tau_B(\mathrm{IA}_n)$. Therefore we obtain $[\overline{K}_{ji}, \overline{K}_{ij}] \equiv 1$, and $[\overline{L}_{ij}, \overline{L}_{ji}] \equiv 1$ in $\tau_B(\mathrm{IA}_n)/\mathcal{Y}_n$. □

Lemma 4.3. *For $1 \leq i \neq j \neq k \leq n$, $[\overline{L}_{ij}, \overline{L}_{ik}]$, $[\overline{L}_{ij}, \overline{L}_{jk}] \in \mathcal{Y}_n$.*

Proof. It suffices to consider the case where $1 \leq i, j, k \leq n - 1$. By a direct computation, we see

$$[\overline{L}_{ij}, \overline{L}_{ik}] = [\overline{L}_{ij}, \overline{L}_{in}][\overline{L}_{in}, \overline{L}_{ik}] \in \mathcal{Y}_n. \tag{3}$$

Furthermore, we have

$$[\overline{L}_{ij}, \overline{L}_{jk}^{-1}] = [\overline{L}_{ij}, \overline{L}_{ik}] \in \mathcal{Y}_n. \tag{4}$$

Hence $[\overline{L}_{ij}, \overline{L}_{jk}] \in \mathcal{Y}_n$. □

Then we have:

Lemma 4.4. *For any $n \geq 2$, $\mathcal{B}_n'(2) \subset \mathcal{Y}_n$.*

Proof. Since $\mathcal{B}_n'(2)$ is generated by commutators $[\overline{L}_{ij}, \overline{L}_{kl}]$ as a normal subgroup, and since $\mathcal{Y}_n$ is a normal subgroup of $\tau_B(\mathrm{IA}_n)$, it suffices to show that $[\overline{L}_{ij}, \overline{L}_{kl}]$ is contained in $\mathcal{Y}_n$ for any $1 \leq i, j, k, l \leq n$. Set $X := [\overline{L}_{ij}, \overline{L}_{kl}]$ and $N := \sharp\{i, j, k, l\}$. If $N = 4$, $X = 1$. For $N = 2$ or 3, we see $X \in \mathcal{Y}_n$ from Lemmas 4.2 and 4.3. □

Here we remark that the quotient group of $\tau_B(\mathrm{IA}_n)$ by $\mathcal{Y}_n$ is given by

Proposition 4.1. *For $n \geq 2$, $\tau_B(\mathrm{IA}_n)/\mathcal{Y}_n \cong H_1(\tau_B(\mathrm{IA}_{n-1}), \mathbf{Z})$.*

Proof. For any $\sigma \in \mathrm{IA}_{n-1}$, defining an automorphism of F_n by

$$x_i \mapsto \begin{cases} x_i^{\sigma}, & 1 \leq i \leq n-1, \\ x_n, & i = n, \end{cases}$$

we obtain an injective homomorphism $\mathrm{IA}_{n-1} \to \mathrm{IA}_n$. This homomorphism induces an injective homomorphism $\eta : \tau_B(\mathrm{IA}_{n-1}) \to \tau_B(\mathrm{IA}_n)$ defined by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Composing η with the natural projection $\tau_B(\mathrm{IA}_n) \to \tau_B(\mathrm{IA}_n)/\mathcal{Y}_n$, we obtain a homomorphism $\overline{\eta} : \tau_B(\mathrm{IA}_{n-1}) \to \tau_B(\mathrm{IA}_n)/\mathcal{Y}_n$. By the definition of $\mathcal{Y}_n$, $\overline{\eta}$ is surjective. Furthermore, since the target of $\overline{\eta}$ is an abelian group, $\overline{\eta}$ induces a homomorphism $\overline{\eta}_1 : H_1(\tau_B(\mathrm{IA}_{n-1}), \mathbf{Z}) \to \tau_B(\mathrm{IA}_n)/\mathcal{Y}_n$.

Next, we show that $\overline{\eta}_1$ is injective. Recall that $H_1(\tau_B(\mathrm{IA}_{n-1}), \mathbf{Z})$ is a free abelian group on (the coset classes of) $\overline{L}_{ij}$ for $1 \le i \ne j \le n-1$. For an element

$$\gamma = \overline{L}_{12}^{e_{12}} \cdots \overline{L}_{1n-1}^{e_{1n-1}} \cdots \overline{L}_{n-11}^{e_{n-11}} \cdots \overline{L}_{n-1n-2}^{e_{n-1n-2}} \in H_1(\tau_B(\mathrm{IA}_{n-1}), \mathbf{Z}), \quad e_{ij} \in \mathbf{Z},$$

let $\overline{\eta}_1(\gamma) = 1$. Then $\overline{\eta}_1(\gamma) \in \mathcal{Y}_n$. Therefore, considering the image of the natural projection $\tau_B(\mathrm{IA}_n) \to H_1(\tau_B(\mathrm{IA}_n), \mathbf{Z})$ restricted to $\mathcal{Y}_n$, we obtain an equation

$$\overline{L}_{12}^{e_{12}} \cdots \overline{L}_{1n-1}^{e_{1n-1}} \cdots \overline{L}_{n-11}^{e_{n-11}} \cdots \overline{L}_{n-1n-2}^{e_{n-1n-2}} = \overline{L}_{n1}^{e_{n1}} \cdots \overline{L}_{nn-1}^{e_{nn-1}}$$

for some $e_{ni} \in \mathbf{Z}$ in $H_1(\tau_B(\mathrm{IA}_n), \mathbf{Z})$. On the other hand, since $H_1(\tau_B(\mathrm{IA}_n), \mathbf{Z})$ is a free abelian group on $\overline{L}_{ij}$ for $1 \le i \ne j \le n$, we obtain that $e_{ij} = 0$ for any $1 \le i \ne j \le n$. Hence $\overline{\eta}_1$ is injective. This completes the proof of Proposition 4.1. □

Next we show that $\mathcal{Y}_n'(k)$ coincides with $\mathcal{B}_n'(k)$ for any $k \ge 2$.

Theorem 4.2. *For any $n \ge 2$ and $k \ge 2$, $\mathcal{Y}_n'(k) = \mathcal{B}_n'(k)$.*

Proof. We prove $\mathcal{Y}_n'(k) \supset \mathcal{B}_n'(k)$ by induction on k. For $k = 2$, we show that $[\overline{L}_{ij}, \overline{L}_{kl}] \in \mathcal{Y}_n'(2)$ for any $1 \le i, j, k, l \le n$. Set $N := \sharp\{i, j, k, l\}$. If $N = 4$, $X = 1$. For $N = 2$, it suffices to show that $[\overline{L}_{ij}, \overline{L}_{ji}] \in \mathcal{Y}_n'(2)$. If $i = n$ or $j = n$, it is clear. From (2), for $i, j \ne n$, we have

$$1 = [\overline{K}_{ji}\overline{K}_{ni}, \overline{K}_{ij}] = [\overline{K}_{ji}, [\overline{K}_{ni}, \overline{K}_{ij}]][\overline{K}_{ni}, \overline{K}_{ij}][\overline{K}_{ji}, \overline{K}_{ij}] \in \mathcal{Y}_n/\mathcal{Y}_n'(2)$$

Since $\tau_B(\mathrm{IA}_n)$ acts on $\mathcal{Y}_n/\mathcal{Y}_n'(2)$ trivially by Lemma 4.1,

$$[\overline{K}_{ni}, \overline{K}_{ij}] = \overline{K}_{ni}(\overline{K}_{ij}, \overline{K}_{ni}^{-1}\overline{K}_{ij}^{-1}) = \overline{K}_{ni}\overline{K}_{ni}^{-1} = 1 \in \mathcal{Y}_n/\mathcal{Y}_n'(2).$$

Therefore $[\overline{K}_{ji}, \overline{K}_{ij}] \in \mathcal{Y}_n'(2)$, and hence $[\overline{L}_{ij}, \overline{L}_{ji}] = [\overline{K}_{ij}^{-1}, \overline{K}_{ji}^{-1}] \in \mathcal{Y}_n'(2)$. For $N = 3$, it suffices to show that $[\overline{L}_{ij}, \overline{L}_{ik}]$, $[\overline{L}_{ij}, \overline{L}_{jk}] \in \mathcal{Y}_n'(2)$ for $1 \le i \ne j \ne k \le n$. For $[\overline{L}_{ij}, \overline{L}_{ik}]$, if $i = n$, it is clear. If $k = n$, we see $[\overline{L}_{ij}, \overline{L}_{in}] = [\overline{L}_{nj}^{-1}, \overline{L}_{in}] \in \mathcal{Y}_n'(2)$. Similarly $[\overline{L}_{in}, \overline{L}_{ik}] \in \mathcal{Y}_n'(2)$. If $i, j, k \ne n$, then from (3), $[\overline{L}_{ij}, \overline{L}_{ik}] \in \mathcal{Y}_n'(2)$. Finally, from (4), we obtain $[\overline{L}_{ij}, \overline{L}_{jk}] \in \mathcal{Y}_n'(2)$.

For $k \ge 2$, suppose $\mathcal{Y}_n'(m) \supset \mathcal{B}_n'(m)$ for $2 \le m \le k$. Now, $\mathcal{B}_n'(k+1)$ is generated by commutators of type

$$[\sigma, \tau], \quad \sigma \in \mathcal{B}_n'(m_1), \tau \in \mathcal{B}_n'(m_2), m_1 + m_2 = k + 1.$$

We may assume that $m_1 \ge m_2$. If $m_2 \ge 2$, since $\mathcal{B}_n'(m_l) = \mathcal{Y}_n'(m_l)$ for $l = 1$ and 2 by the inductive hypothesis, we see $[\sigma, \tau] \in \mathcal{Y}_n'(k+1)$. Let $m_2 = 1$. Then $\sigma \in \mathcal{B}_n'(m_1) = \mathcal{Y}_n'(m_1)$. Since $\tau_B(\mathrm{IA}_n)$ acts on $\mathcal{Y}_n/\mathcal{Y}_n'(2)$ trivially, the natural action of $\tau_B(\mathrm{IA}_n)$ on $\mathrm{gr}^m(\mathcal{Y}_n')$ by conjugation is also trivial for any $m \ge 2$. Hence

$$[\sigma, \tau] = \sigma(\tau\sigma^{-1}\tau^{-1}) = \sigma\sigma^{-1} = 1 \in \mathrm{gr}^k(\mathcal{Y}_n').$$

This shows that $[\sigma, \tau] \in \mathcal{Y}_n'(k+1)$. This completes the proof of Theorem 4.2. □

Next we determine $\mathrm{gr}^2(\mathcal{B}_n')$ using the homomorphism $\xi_{2,1}$.

Theorem 4.3. *For $n \ge 2$, $\mathrm{gr}^2(\mathcal{B}_n') \cong \mathbf{Z}^{\oplus(n^2-n-1)}$.*

Proof. Since we have a surjective homomorphism $\xi_{2,1} : \mathrm{gr}^2(\mathcal{B}_n') \to \mathbf{Z}^{\oplus n^2-n-1}$, it suffices to show that $\mathrm{gr}^2(\mathcal{B}_n')$ is generated by $n^2 - n - 1$ elements. By Theorem 4.2, $\mathrm{gr}^2(\mathcal{B}_n') = \mathcal{Y}_n'(2)/\mathcal{Y}_n'(3)$ is generated by

$$\{[\overline{L}_{in}, \overline{L}_{jn}] \mid 1 \le i, j \le n-1\} \cup \{[\overline{L}_{in}, \overline{L}_{nj}] \mid 1 \le i, j \le n-1\}$$
$$\cup \{[\overline{L}_{ni}, \overline{L}_{nj}] \mid 1 \le j \le i \le n-1\}.$$

On the other hand, we see $[\overline{L}_{in}, \overline{L}_{jn}] = 1$ and

$$[\overline{L}_{ni}, \overline{L}_{nj}] = [\overline{L}_{ni}, \overline{L}_{nn-1}][\overline{L}_{nn-1}, \overline{L}_{nj}]$$

for $1 \le i, j \le n-1$. Hence

$$\{[\overline{L}_{in}, \overline{L}_{nj}] \mid 1 \le i, j \le n-1\} \cup \{[\overline{L}_{nn-1}, \overline{L}_{nj}] \mid 1 \le j \le i \le n-1\}$$

generates $\mathrm{gr}^2(\mathcal{B}_n')$. The number of the set above is just $n^2 - n - 1$. This completes the proof of Theorem 4.3. □

As a corollary, we obtain

Corollary 4.1. *For $n \ge 2$, $\mathcal{B}_n(3) = \mathcal{B}_n'(3)$.*

Proof. Since the isomorphism $\xi_{2,1} : \mathrm{gr}^2(\mathcal{B}_n') \to \mathbf{Z}^{\oplus(n^2-n-1)}$ factors through $\mathrm{gr}^2(\mathcal{B}_n)$, a natural homomorphism $\mathrm{gr}^2(\mathcal{B}_n') \to \mathrm{gr}^2(\mathcal{B}_n)$ is an isomorphism. Hence $\mathcal{B}_n(3) = \mathcal{B}_n'(3)$. □

By Pettet [12], $\mathcal{A}_n'(3)$ has a finite index in $\mathcal{A}_n(3)$. From Corollary 4.1, we see that if $\mathcal{A}_n'(3) \ne \mathcal{A}_n(3)$, the difference between them is contained in the kernel of τ_B.

5. The cup product

In this section we determine the image of the cup product

$$\cup : \Lambda^2 H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z}).$$

First, we consider an interpretation of the second cohomology group $H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z})$.

Let F be a free group of rank $n(n-1)$ on $\{\overline{L}_{ij} \mid 1 \leq i \neq j \leq n\}$. Let $\varphi : F \to \tau_B(\mathrm{IA}_n)$ be a natural surjection and R the kernel of φ. Then we have a minimal presentation of $\tau_B(\mathrm{IA}_n)$

$$1 \to R \to F \xrightarrow{\varphi} \tau_B(\mathrm{IA}_n) \to 1. \tag{5}$$

The word "minimal" means that the number of generators is minimal among any presentation of $\tau_B(\mathrm{IA}_n)$. Since the abelianization of $\tau_B(\mathrm{IA}_n)$ is a free abelian group with basis $\{\overline{L}_{ij} \mid 1 \leq i \neq j \leq n\}$ by Theorem 4.1, the induced homomorphism

$$\varphi^* : H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^1(F, \mathbf{Z})$$

is an isomorphism. Hence considering the cohomological five-term exact sequence

$$0 \to H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^1(F, \mathbf{Z}) \to H^1(R, \mathbf{Z})^{\tau_B(\mathrm{IA}_n)} \to H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^2(F, \mathbf{Z}) = 0$$

of (5), we obtain an isomorphism

$$H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \cong H^1(R, \mathbf{Z})^{\tau_B(\mathrm{IA}_n)}.$$

To study the abelian group $H^1(R, \mathbf{Z})^{\tau_B(\mathrm{IA}_n)}$, we consider a descending filtration of R. Let $\Gamma_F(k)$ be the lower central series of F and $\mathcal{L}_F(k) := \Gamma_F(k)/\Gamma_F(k+1)$ for $k \geq 1$. Set $R_k := R \cap \Gamma_F(k)$ and $\overline{R}_k := R/R_k$ for $k \geq 1$. Then $R_k = R$ for $1 \leq k \leq 2$. The natural projection $R \to \overline{R}_{k+1}$ induces an injective homomorphism

$$\psi^k : H^1(\overline{R}_{k+1}, \mathbf{Z})^{\tau_B(\mathrm{IA}_n)} \to H^1(R, \mathbf{Z})^{\tau_B(\mathrm{IA}_n)}.$$

Hence we can consider $H^1(\overline{R}_{k+1}, \mathbf{Z})^{\tau_B(\mathrm{IA}_n)}$ as a subgroup of $H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z})$. In the following, we study the case where $k = 2$. In this case, we remark that $H^1(\overline{R}_3, \mathbf{Z})^{\tau_B(\mathrm{IA}_n)} = H^1(\overline{R}_3, \mathbf{Z})$ since $\tau_B(\mathrm{IA}_n)$ acts on $\overline{R}_3$ trivially. Then we have:

Proposition 5.1. *The image of the cup product*

$$\cup : \Lambda^2 H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z})$$

is $H^1(\overline{R}_3, \mathbf{Z})$.

Proof. First, considering the cohomological five-term exact sequence of

$$1 \to \mathcal{B}_n'(2) \to \tau_B(\mathrm{IA}_n) \xrightarrow{p} \tau_B(\mathrm{IA}_n)^{\mathrm{ab}} \to 1, \tag{6}$$

we have

$$\begin{aligned}0 &\to H^1(\tau_B(\mathrm{IA}_n)^{\mathrm{ab}}, \mathbf{Z}) \to H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^1(\mathcal{B}_n'(2), \mathbf{Z})^{\tau_B(\mathrm{IA}_n)} \\ &\to H^2(\tau_B(\mathrm{IA}_n)^{\mathrm{ab}}, \mathbf{Z}) \to H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z}).\end{aligned}$$

Since $H^1(\tau_B(\mathrm{IA}_n)^{\mathrm{ab}}, \mathbf{Z}) \cong H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z})$, and $H^1(\mathcal{B}_n'(2), \mathbf{Z})^{\tau_B(\mathrm{IA}_n)} = H^1(\mathrm{gr}^2(\mathcal{B}_n'), \mathbf{Z})$, we obtain an exact sequence

$$0 \to H^1(\mathrm{gr}^2(\mathcal{B}_n'), \mathbf{Z}) \to H^2(\tau_B(\mathrm{IA}_n)^{\mathrm{ab}}, \mathbf{Z}) \to H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z}).$$

Since $H_1(\tau_B(\mathrm{IA}_n), \mathbf{Z})$ is a free abelian group of finite rank,

$$H^2(\tau_B(\mathrm{IA}_n)^{\mathrm{ab}}, \mathbf{Z}) \cong \Lambda^2 H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z}),$$

and $p^* : H^2(\tau_B(\mathrm{IA}_n)^{\mathrm{ab}}, \mathbf{Z}) \to H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z})$ is considered as the cup product $\cup : \Lambda^2 H^1(\tau_B(\mathrm{IA}_n), \mathbf{Z}) \to H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z})$.

On the other hand, considering the five-term exact sequence of

$$0 \to R/R_3 \to \mathcal{L}_F(2) \xrightarrow{\varphi_2} \mathrm{gr}^2(\mathcal{B}_n') \to 0,$$

we have

$$0 \to H^1(\mathrm{gr}^2(\mathcal{B}_n'), \mathbf{Z}) \to H^1(\mathcal{L}_F(2), \mathbf{Z}) \to H^1(\overline{R}_3, \mathbf{Z})^{\mathcal{L}_F(2)} \to H^2(\mathrm{gr}^2(\mathcal{B}_n'), \mathbf{Z}) \to H^2(\mathcal{L}_F(2), \mathbf{Z}).$$

Since $\mathcal{L}_F(2)$ acts on $\overline{R}_3$ trivially, we have $H^1(\overline{R}_3, \mathbf{Z})^{\mathcal{L}_F(2)} = H^1(\overline{R}_3, \mathbf{Z})$. Furthermore, since $\mathrm{gr}^2(\mathcal{B}_n')$ is a free abelian group by Theorem 4.3, the second homomorphism $H^1(\mathcal{L}_F(2), \mathbf{Z}) \to H^1(\overline{R}_3, \mathbf{Z})$ is surjective. Then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathrm{gr}^2(\mathcal{B}_n'), \mathbf{Z}) \arrow[r, "\mathrm{tg}"] \arrow[d, equal] & H^2(\tau_B(\mathrm{IA}_n)^{\mathrm{ab}}, \mathbf{Z}) \arrow[r, "p^*"] \arrow[d, "\mu"] & H^2(\tau_B(\mathrm{IA}_n), \mathbf{Z}) & \\
0 \arrow[r] & H^1(\mathrm{gr}^2(\mathcal{B}_n'), \mathbf{Z}) \arrow[r, "\varphi_2^*"] & H^1(\mathcal{L}_F(2), \mathbf{Z}) \arrow[r] & H^1(\overline{R}_3, \mathbf{Z}) \arrow[r] & 0
\end{tikzcd}$$

where tg is the transgression and μ is a natural isomorphism. Hence we obtain $\mathrm{Im}(\cup) \cong \mathrm{Im}(p^*)$. This completes the proof of Proposition 5.1. □

Since $\mathcal{L}_F(2)$ is a free abelian group of rank $n(n-1)(n^2-n-1)/2$, we have

$$R/R_3 \cong \mathbf{Z}^{\oplus(n-2)(n+1)(n^2-n-1)/2}. \tag{7}$$

This shows

Theorem 5.1. *For $n \geq 2$, $\mathrm{Im}(\cup) \cong \mathbf{Z}^{\oplus(n-2)(n+1)(n^2-n-1)/2}$.*

In general, any presentation for $\tau_B(\mathrm{IA}_n)$ is not known. From the result (7), any normally generating set of R in F must have $(n-2)(n+1)(n^2-n-1)/2$ elements, and hence we see that $(n-2)(n+1)(n^2-n-1)/2$ is a lower bound on the number of relations among the generators $\overline{L}_{ij}$ of $\tau_B(\mathrm{IA}_n)$.

6. The case $n = 2$

In this section, we completely determine the structures of $\mathrm{gr}^k(\mathcal{B}'_2)$ and $\mathrm{gr}^k(\mathcal{B}_2)$ for any $k \geq 1$. Recall that $\mathrm{IA}_2 = \mathrm{Inn}\, F_2$ is generated by K_{21} and K_{12}. For the convenience, set $\iota_1 := K_{21}$ and $\iota_2 := K_{12}$. We remark that from Theorem 4.1, the abelianization of $\tau_B(\mathrm{IA}_2)$ is a free abelian group of rank 2 generated by ι_1 and ι_2.

Theorem 6.1. *For any $k \geq 2$, $\mathrm{gr}^k(\mathcal{B}'_2) \cong \mathbf{Z}$.*

Proof. The abelian group $\mathrm{gr}^k(\mathcal{B}'_2)$ is generated by the images of

$$[\iota_1, \iota_2, \iota_{i_1}, \ldots, \iota_{i_{k-2}}], \quad i_j = 1, 2$$

by τ_B. Let p and q be the number of ι_1 and ι_2 which appear in the component of $[\iota_1, \iota_2, \iota_{i_1}, \ldots, \iota_{i_{k-2}}]$ respectively. Then we have

$$\tau_G([\iota_1, \iota_2, \iota_{i_1}, \ldots, \iota_{i_{k-2}}]) = \begin{pmatrix} 1 + (1-s_1)^{1+p}(1-s_2)^{1+q} & -(1-s_1)^{2+p}(1-s_2)^{q} \\ (1-s_1)^{p}(1-s_2)^{2+q} & 1-(1-s_1)^{1+p}(1-s_2)^{1+q} \end{pmatrix},$$

and hence,

$$\tau_B([\iota_1, \iota_2, \iota_{i_1}, \ldots, \iota_{i_{k-2}}]) = \begin{pmatrix} 1 + (1-s)^{k} & -(1-s)^{k} \\ (1-s)^{k} & 1-(1-s)^{k} \end{pmatrix}.$$

Therefore, we can reduce the generators $\tau_B([\iota_1, \iota_2, \iota_{i_1}, \ldots, \iota_{i_{k-2}}])$ except for

$$\tau_B([\iota_1, \iota_2, \iota_2, \ldots, \iota_2]).$$

Namely, $\mathrm{gr}^k(\mathcal{B}'_2)$ is generated by only one element. On the other hand, we have a surjective homomorphism $\xi_{k,k} : \mathrm{gr}^k(\mathcal{B}'_2) \to \mathbf{Z}$. Therefore we conclude that $\mathrm{gr}^k(\mathcal{B}'_2) \cong Z$. □

Since $\mathcal{A}_2(k) = \mathcal{A}'_2(k)$ for any $k \geq 1$ due to Andreadakis [1], we obtain

Corollary 6.1. *For any $k \geq 2$, $\mathrm{gr}^k(\mathcal{B}_2) \cong \mathbf{Z}$.*

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